7.1.1 Properties of Linear Noise

- Linearly uncorrelated to signal:

$$
\begin{aligned}
\langle n s^\dagger \rangle_{(d,s)} &= \langle (d - R s) s^\dagger \rangle_{(d,s)} \\
&= \langle d s^\dagger \rangle - R \langle s s^\dagger \rangle \\
& \langle d s^\dagger \rangle - \langle d s^\dagger \rangle_{(d,s)} \langle s s^\dagger \rangle_{(d,s)}^{-1} \langle s s^\dagger \rangle_{(d,s)} \\
&= \langle d s^\dagger \rangle - \langle d s^\dagger \rangle_{(d,s)} = 0
\end{aligned}
$$

- linear noise auto-correlation:

$$
\begin{aligned}
\langle n n^\dagger \rangle_{(d,s)} &= \langle (d - R s)(d - R s)^\dagger \rangle_{(d,s)} \\
&= \langle d d^\dagger \rangle_{(d,s)} - \langle d s^\dagger \rangle_{(d,s)} R^\dagger - R \langle s d^\dagger \rangle + R \langle s s^\dagger \rangle R^\dagger \\
&= (R S R^\dagger + N) - (R S R^\dagger) - (R S R^\dagger) + (R S R^\dagger) \\
&= N
\end{aligned}
$$

Example: Noisless, Non-Linear Data

- $s \in \mathbb{R}$
- $\mathcal{P}(s) = \mathcal{G}(s, \sigma^2)$
- $d = f(s) = s^3$

Moments:

$$\begin{aligned}\langle s\, s^\dagger \rangle_{(s)} &= \sigma^2 \\ \langle d\, s^\dagger \rangle_{(d,s)} &= \langle s^4 \rangle_{(s)} = 3\,\sigma^4 \\ \langle d\, d^\dagger \rangle_{(d,s)} &= \langle s^6 \rangle_{(s)} = \frac{6!}{2^3 3!}\,\sigma^6 = 15\,\sigma^6\end{aligned}$$

Linear response:

$$R = \langle d\, s^\dagger \rangle_{(d,s)} \langle s\, s^\dagger \rangle^{-1}_{(d,s)} = 3\,\sigma^2$$

Example: Noisless, Non-Linear Data

Noise covariance:

$$N = \langle d\, d^\dagger \rangle - \langle d\, s^\dagger \rangle \langle s\, s^\dagger \rangle^{-1} \langle s\, d^\dagger \rangle = 15\,\sigma^6 - 3\,\sigma^4 \times \sigma^{-2} \times 3\,\sigma^4 = 6\,\sigma^6$$

Optimal linear filter:

$$F_{\mathrm{L}} = \langle s\, d^\dagger \rangle_{(d,s)} \langle d\, d^\dagger \rangle^{-1}_{(d,s)} = 3\,\sigma^4 / (15\,\sigma^6) = \frac{1}{5}\,\sigma^{-2}$$

Reconstruction Error:

$$\langle (s - F_{\mathrm{L}} d)^2 \rangle = \langle s^2 \rangle - 2\,F_{\mathrm{L}} \langle s^4 \rangle + F_{\mathrm{L}}^2 \langle s^6 \rangle = \left(1 - \frac{6}{5} + \frac{15}{25} \right) \sigma^2 = \frac{2}{5}\,\sigma^2$$

Maximum Entropy Perspective

- known covariances: $\langle dd^\dagger \rangle_{(d,s)}$, $\langle ss^\dagger \rangle_{(d,s)}$, $\langle ds^\dagger \rangle_{(d,s)}$
- MaxEnt models $\mathcal{P}(d,\, s)$ as Gaussian via moment constraints

$\Rightarrow$ Optimal signal estimate given a priori only second moments is the Wiener filter.

7.2 Symmetry between Filter and Response

$$
\begin{aligned}
\mathcal{P}(n, s) &= \mathcal{G}(n, N)\mathcal{G}(s, S) \\
\mathcal{P}(d, s) &= \int dn\, \mathcal{P}(d, n, s) \\
&= \int dn\, \mathcal{P}(d|n, s)\, \mathcal{P}(n, s) \\
&= \int dn\, \delta(d - (Rs + n))\mathcal{G}(n, N)\mathcal{G}(s, S) \\
&= \mathcal{G}(d - Rs, N)\mathcal{G}(s, S)
\end{aligned}
$$

- signal estimate:

$$
\langle s \rangle_{(s|d)} = F_W\, d = F_L d = \langle s d^\dagger \rangle_{(d,s)} \langle d d^\dagger \rangle^{-1}_{(d,s)} d
$$

- signal response:

$$
\langle d \rangle_{(d|s)} = \langle Rs + n \rangle_{(n|s)} = Rs + \underbrace{\langle n \rangle_{(n)}}_{=0} = Rs = \langle d^\dagger s \rangle_{(d,s)} \langle s s^\dagger \rangle^{-1}_{(d,s)} s
$$

7.2 Symmetry between Filter and Response

$$\begin{aligned}\langle s\rangle_{(s|d)} &= \langle s d^\dagger\rangle_{(d,s)}\langle d d^\dagger\rangle^{-1}_{(d,s)}\, d\\ \langle d\rangle_{(d|s)} &= \langle d^\dagger s\rangle_{(d,s)}\langle s s^\dagger\rangle^{-1}_{(d,s)}\, s\end{aligned}$$

Symmetry between filter and response by exchange of data and signal:

$$\begin{aligned}\text{signal estimate} &\mathrel{\hat{=}} \text{data response}\\ \text{data estimate} &\mathrel{\hat{=}} \text{signal response}\end{aligned}$$

Combined Probability Distribution

- combined vector x:

$$x = \begin{pmatrix} d \\ s \end{pmatrix}$$

- combined covariance X:

$$X = \langle x x^{\dagger} \rangle_{(x)} = \begin{pmatrix} \langle d d^{\dagger} \rangle_{(d,\, s)} & \langle d s^{\dagger} \rangle_{(d,\, s)} \\ \langle s d^{\dagger} \rangle_{(d,\, s)} & \langle s s^{\dagger} \rangle_{(d,\, s)} \end{pmatrix}$$

- combined probability distribution:

$$\mathcal{P}(x|\, X) = \mathcal{G}(x,\, X)$$

Combined Probability Distribution

x_a, x_b: subvectors (e.g. $x_a = s$, $x_b = d$)

Mean:

$$m_a := \langle x_a \rangle_{(x_a|x_b)} = X_{ab}(X_{bb})^{-1} x_b$$

Covariance:

$$D_{aa} = \left[\underbrace{X_{aa}^{-1}}_{=S^{-1}} + \underbrace{X_{aa}^{-1} X_{ab}}_{=R^{\dagger}} \underbrace{\left(X_{bb} - X_{ab}^{\dagger} X_{aa}^{-1} X_{ab} \right)^{-1}}_{=N^{-1}} \underbrace{X_{ab}^{\dagger} X_{aa}^{-1}}_{=R} \right]^{-1}$$

Posterior probability distribution $P(x_a|x_b)$:

$$P(x_a|x_b) = \mathcal{G}(x_a - m_a,\, D_{aa})$$

7.3 Response

- R translates between signal and data space
- $R(s)$ is image of signal in data space

generic response:

$$R(s) := \langle d \rangle_{(d|s)}$$

linear response:

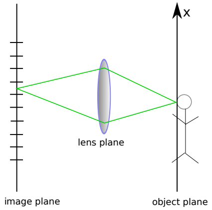
$$R(s) = Rs \text{ with } R = \langle ds^\dagger \rangle_{(d,\,s)} \langle ss^\dagger \rangle^{-1}_{(d,\,s)}$$

7.3.1 Repeated measurement of $s \in \mathbb{R}$

$$R \;:\; \mathbb{R} \to \mathbb{R}^{\mathtt{n}},\; R = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},\; R(s) = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}$$

$$\begin{aligned}
\Rightarrow d_i &= Rs + n_i \\
d &= \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} s + \begin{pmatrix} n_1 \\ \vdots \\ n_n \end{pmatrix} \\
\text{assume } N &= \mathrm{diag}(\sigma_i^2)_{i=1}^n,\; S = \infty \\
m &= Dj = \left(S^{-1} + R^\dagger N^{-1} R\right)^{-1} R^\dagger N^{-1} d \\
&= \left(0 + \sum_{i=1}^{n} \sigma_i^{-2}\right)^{-1} \sum_{i=1}^{n} \sigma_i^{-2} d_i = \langle d_i \rangle_{\sigma_i^{-2}}
\end{aligned}$$

7.3.2 Photography



$$R \quad : \quad C(\mathbb{R}^2) \to \mathbb{R}^m$$

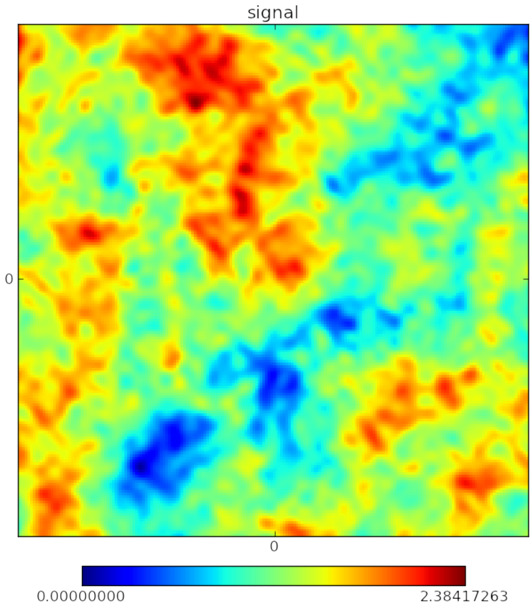
e.g. individual detector:

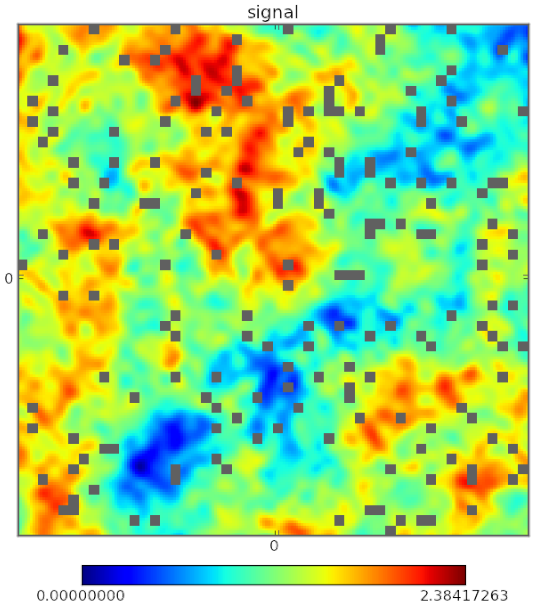
$$R_i : C(\mathbb{R}^2) \to \mathbb{R}$$

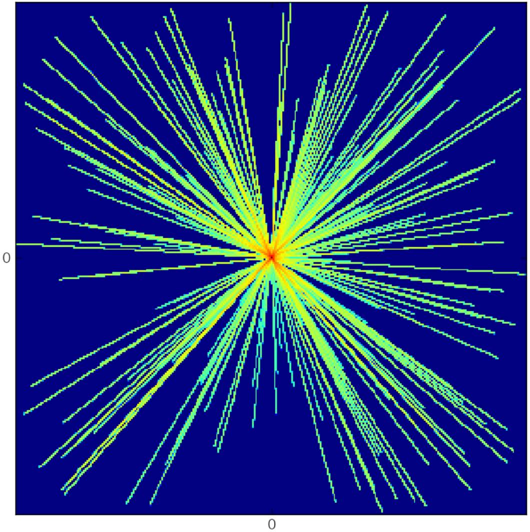
$$d_i = (R\,s + n)_i = \int_{\mathbb{R}^2} d^2x\, R_i(x)\, s(x) + n_i$$

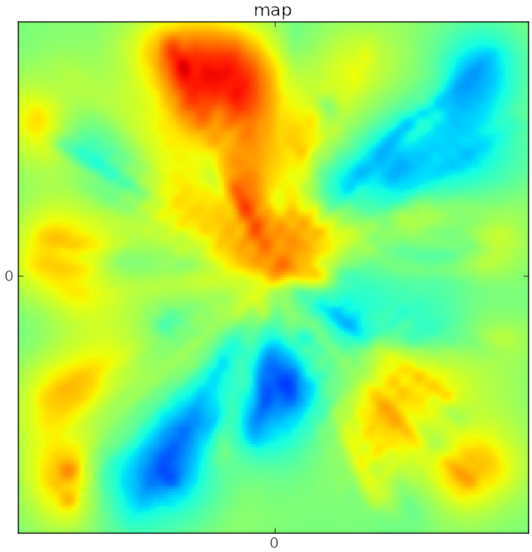
7.3.3 Tomography

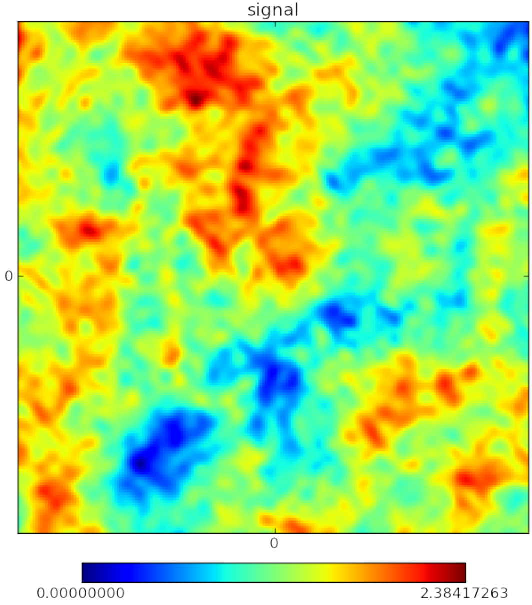
- $\Omega = \mathbb{R}^u$: volume probed
- $a_i \in \Omega$: location of a detector
- $b_i \in \mathcal{S}^{u-1}$: direction
- $x_i(t) = a_i + t\, b_i$: set of rays

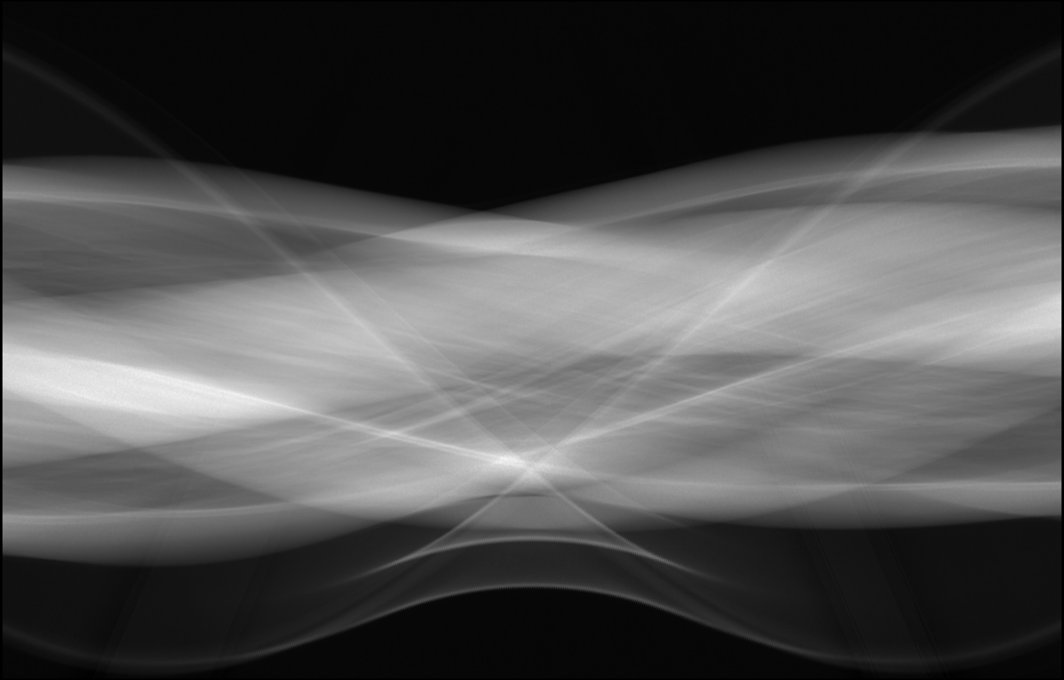

signal
0
0
0.00000000
2.38417263

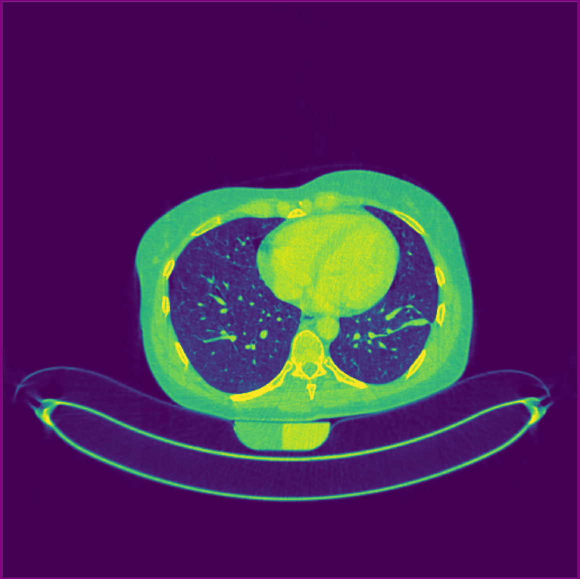

signal
0
0
0.00000000
2.38417263


0
0

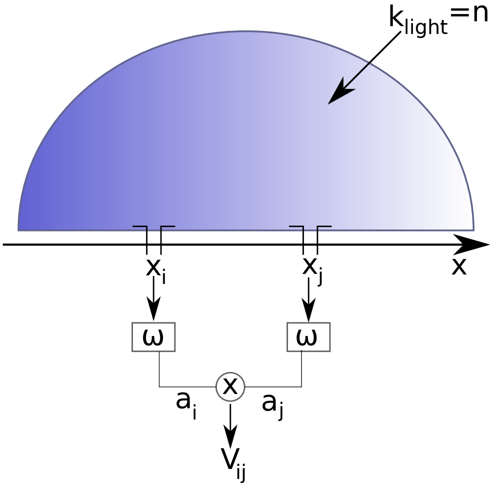

map
0
0


signal
0
0
0.00000000
2.38417263





7.3.4 Interferometry



Measure correlations of incoming waves (EM, gravity, sound, ...) at pairs of different locations for a narrow spectral window with mean frequency ω

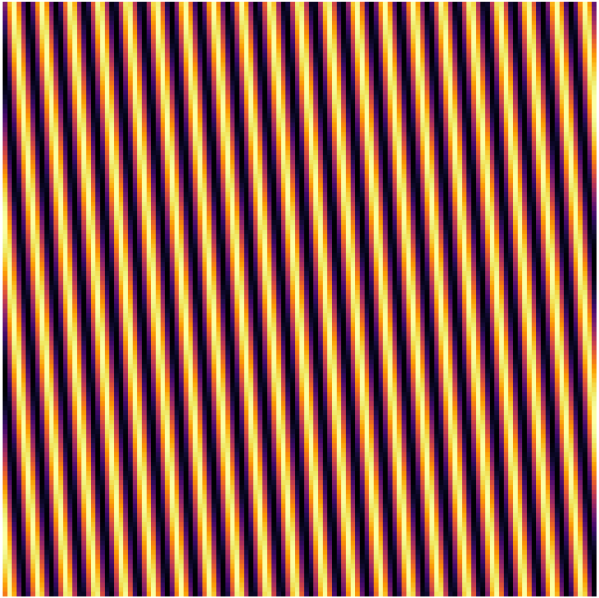
- $s_{\hat{n}}$: sky brightness at sky position $\hat{n}$
- $w(\hat{n}, \vec{x_i}, t)$ = wave from $\hat{n}$ to $\vec{x_i}$
  $= \sqrt{s_{\hat{n}}} \exp[i(\omega t + \varphi(\hat{n},\, t) + \frac{\omega}{c}\hat{n}\vec{x_i})]$
- $a_i = \int d\hat{n}\, w(\hat{n}, \vec{x_i}, t)$: amplitude at antenna i
- $V_{ij} = \langle a_i a_j \rangle_{\text{time average}}$: visibility
- $d_{ij} = V_{ij} + n_{ij}$: measured data
- $\lambda = \frac{c}{\omega}$: wavelength

7.3.4 Interferometry

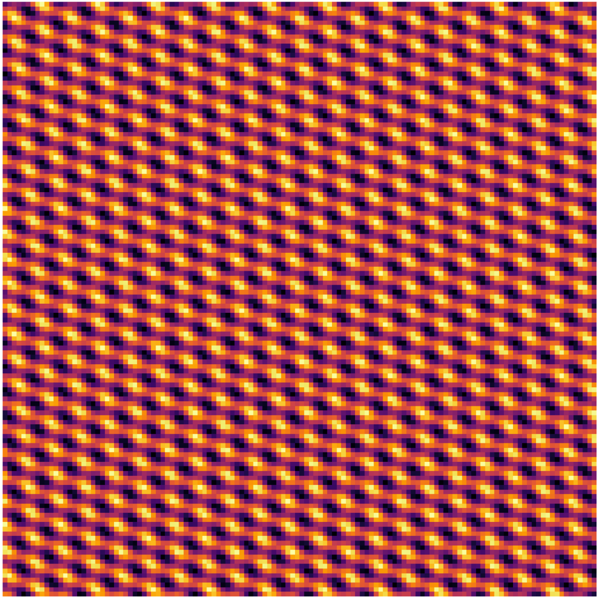
Amplitude: $$a_i = \int_{s^2} d\hat{n}\,\sqrt{s_{\hat{n}}}\,\exp[i(\omega t + \varphi(\hat{n},\,t) + \frac{\omega}{c}\hat{n}\vec{x}_i)]$$

Visibility:
$$\begin{aligned}
V_{ij} &= \langle a_i \overline{a_j}\rangle_t \\
&= \int d\hat{n} \int d\hat{n}'\,\sqrt{s_{\hat{n}} s_{\hat{n}'}}\,\langle e^{i(\omega t - \omega t + \varphi(\hat{n},t) - \varphi(\hat{n}',t))}\rangle_t\, e^{i\frac{\omega}{c}(\hat{n}\vec{x}_i - \hat{n}'\vec{x}_j)} \\
&= \int d\hat{n} \int d\hat{n}'\,\sqrt{s_{\hat{n}} s_{\hat{n}'}}\,\underbrace{\langle e^{i(\varphi(\hat{n},t) - \varphi(\hat{n}',t))}\rangle_t}_{=\delta(\hat{n}-\hat{n}')}\, e^{i\frac{\omega}{c}(\hat{n}\vec{x}_i - \hat{n}'\vec{x}_j)} \\
&= \int d\hat{n}\,\sqrt{s_{\hat{n}} s_{\hat{n}}}\,\exp[i\left(\frac{\vec{x}_i - \vec{x}_j}{\lambda}\right)\cdot\hat{n}] \\
&= \int d\hat{n}\, s_{\hat{n}}\, e^{i\hat{n}\cdot\vec{k}_{ij}} \;\Rightarrow\; R_{(i,j)\,\hat{n}} = e^{i\hat{n}\cdot\vec{k}_{ij}} = F_{\vec{k}_{ij}\,\hat{n}}
\end{aligned}$$

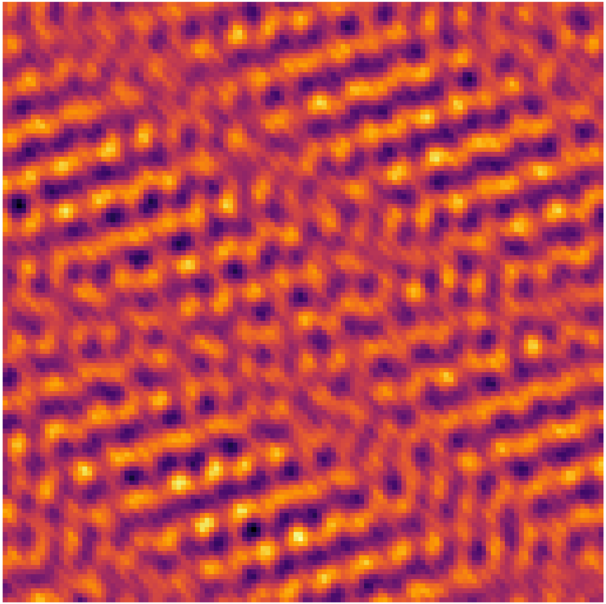
Information source interferometer



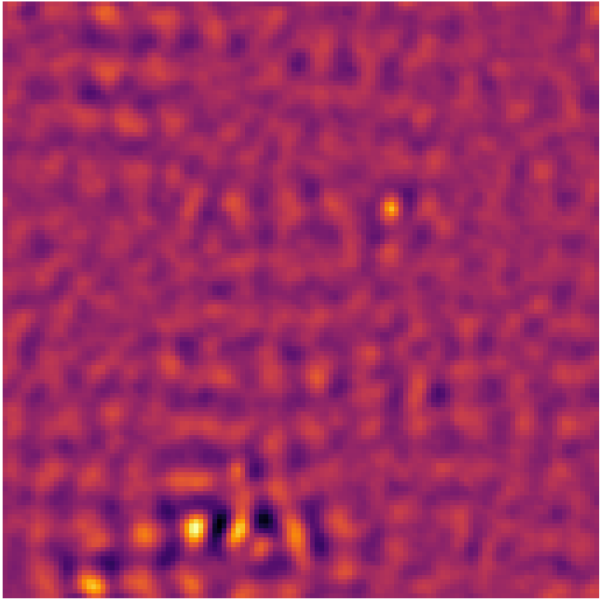
Information source interferometer



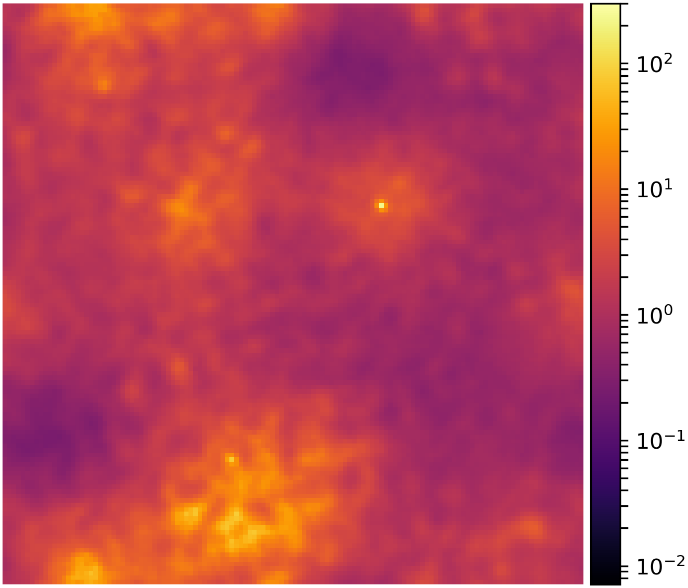
Information source interferometer



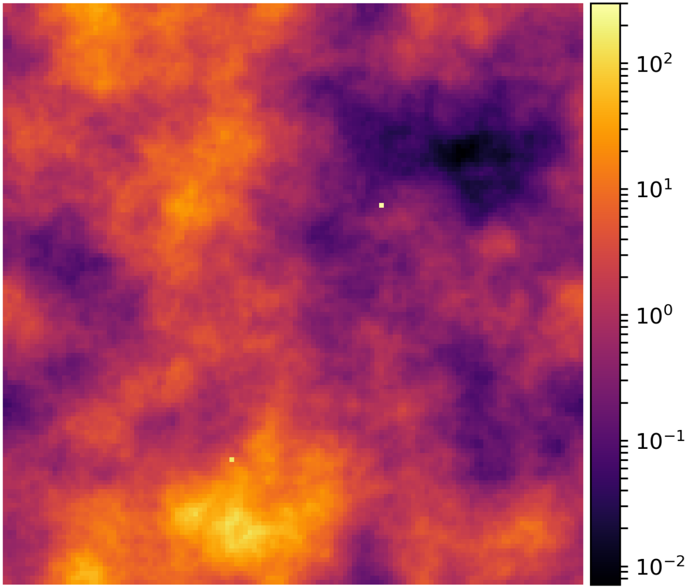
Information source interferometer

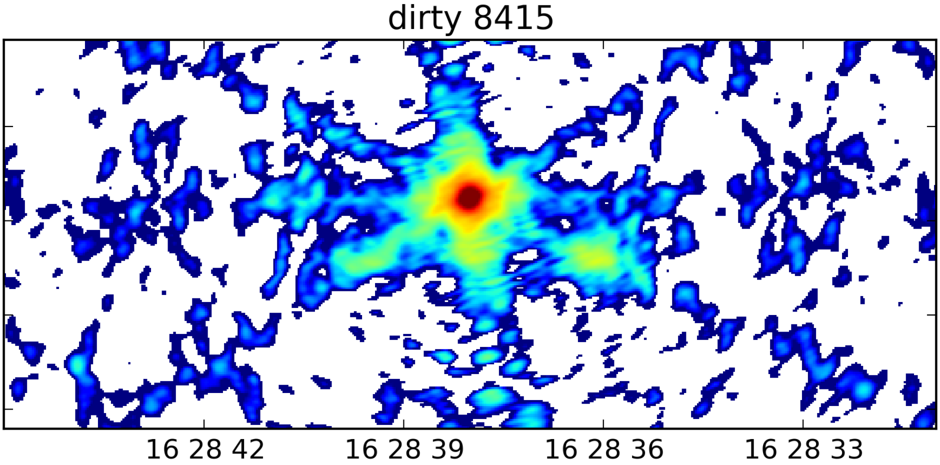


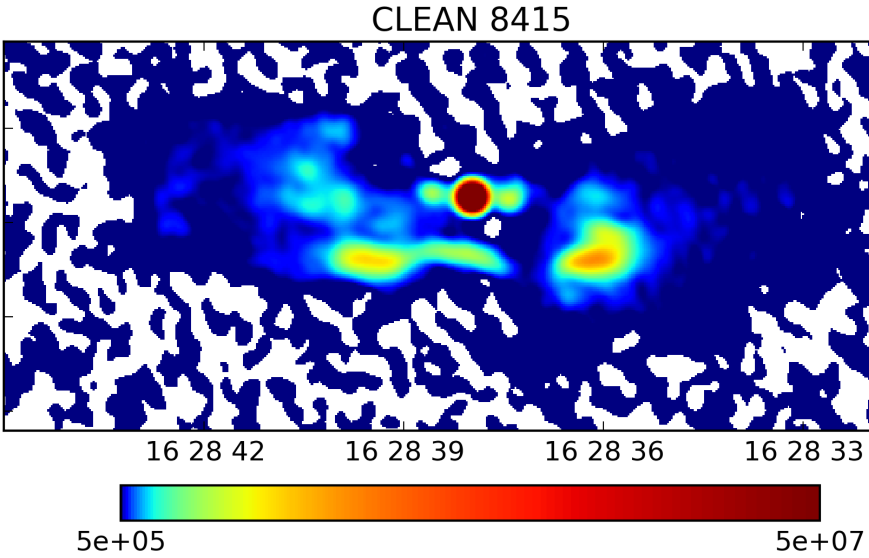
interferometer reconstruction

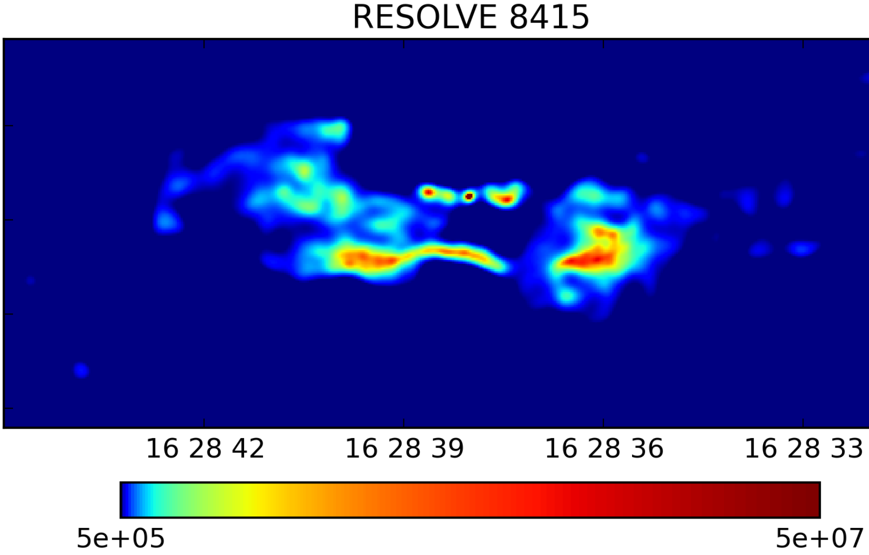


Truth




dirty 8415
16 28 42
16 28 39
16 28 36
16 28 33


CLEAN 8415
16 28 42
16 28 39
16 28 36
16 28 33
5e+05
5e+07


RESOLVE 8415
16 28 42
16 28 39
16 28 36
16 28 33
5e+05
5e+07

8. Gaussian Fields

Multivariate Gaussian (repetition)

$$\mathcal{G}(y, Y) = \frac{1}{\sqrt{|2\pi Y|}} e^{-\frac{1}{2} y^\dagger Y^{-1} y} \text{ with } Y = \langle y y^\dagger \rangle,\ y \in \mathbb{R}^n$$

Now: field with Gaussian statistics $\varphi : \mathbb{R}^u \to \mathbb{R}$

Notation:

- Einstein Summation: $\varphi = \varphi^x e_x$
- contravariant components: $\varphi^x = \varphi(x)$ with $x \in \mathbb{R}^u$
- scalar product: $\psi^\dagger \varphi = \int dx\, \overline{\psi(x)}\, \varphi(x) \equiv \overline{\psi}_x\, \varphi^x$

Contravariant: *Field vector invariant under change of unit system,* $e' = Ae$

$$\varphi' = \varphi'^x e'_x = (A^{-1})^x_y \varphi^y A^z_x e_z = \varphi^y (A^{-1})^x_y A^z_x e_z = \varphi^y \delta^z_y e_z = \varphi^y e_y = \varphi$$

8.Gaussian Fields

$X_{(n)} = \{x_1, ..., x_n\}$: n pixels
$\varphi_{(n)} = (\varphi^{x_1}, \ldots, \varphi^{x_n})^t$: n-dimensional vector of field values

$\Rightarrow$ φ has a Gaussian probability distribution, if for any $X_{(n)} \subset \mathbb{R}^u$

$$\mathcal{P}(\varphi_{(n)}) = \mathcal{G}(\varphi_{(n)}, \Phi_{(n)})$$

with

$$\Phi_{(n)}^{ij} = \langle \varphi_{(n)}^{i} \overline{\varphi_{(n)}^{j}} \rangle = \langle \varphi(x_i) \overline{\varphi(x_j)} \rangle$$

Gaussian Field Distribution

$$\begin{aligned}
\mathcal{G}(\varphi, \Phi) &\equiv \frac{1}{\sqrt{|2\pi\Phi|}} \exp\left(-\frac{1}{2}\varphi^{\dagger}\Phi^{-1}\,\varphi\right) \\
&= \frac{1}{\sqrt{|2\pi\Phi|}} \exp\left(-\frac{1}{2}\overline{\varphi^{x}}\left(\Phi^{-1}\right)_{xy}\,\varphi^{y}\right) \\
&:= \lim_{n\to\infty} \mathcal{G}(\varphi_{(n)}, \Phi_{(n)})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \langle f(\varphi)\rangle_{(\varphi|\Phi)} &= \int \mathcal{D}\varphi\, \mathcal{P}(\varphi|\Phi) f(\varphi) \\
&:= \lim_{n\to\infty} \left[\prod_{i=1}^{n} \int d\varphi^{i}_{(n)}\right] \mathcal{G}(\varphi_{(n)}, \Phi_{(n)}) f(\varphi_{(n)})
\end{aligned}$$

8.1 Field Theory

Scalar Product

discrete: $j^\dagger \varphi = \overline{j}_i \varphi^i$ $\Rightarrow$ continuous: $j^\dagger \varphi = \int dx\, \overline{j(x)}\, \varphi(x) =: \overline{j}_x \varphi^x$

Derivative

discrete: $\partial_{\varphi^i} j^\dagger \varphi = \partial_{\varphi^i} \overline{j}_i \varphi^i = \overline{j}_i \Rightarrow$ continuous:
$\partial_{\varphi^x} j^\dagger \varphi = \frac{\delta}{\delta \varphi^x} \int dx'\, \overline{j_{x'}}\, \varphi^{x'} = \overline{j}_x$ $\Rightarrow$ $\partial_\varphi j^\dagger \varphi = \bar{j}$

Normalisation Factors

discrete: $|\Phi| = \prod_{i=1}^n \lambda_i \Rightarrow$ continuous: $|\Phi| = \lim_{n\to\infty} \prod_{i=1}^n \lambda_i$

Covariance Matrix

discrete: $\Phi^{ij} = \langle \varphi^i \bar{\varphi}^j \rangle$ $\Rightarrow$ continuous: $\Phi^{xy} = \langle \varphi^x \bar{\varphi}^y \rangle_{(\varphi)} = \left(\langle \varphi \varphi^\dagger \rangle_{(\varphi)} \right)^{xy}$

Inverse Covariance

discrete: $\Phi^{-1}\Phi = \mathbb{1}$ $\Rightarrow$ continuous: $\int dy\, \Phi^{-1}_{xy} \Phi^{yz} = \mathbb{1}_x^z = \delta(x - z)$

Wick Theorem

$\langle \varphi^x \varphi^y \varphi^z \varphi^w \rangle_{\mathcal{G}(\varphi,\, \Phi)} = \Phi^{xy}\Phi^{zw} + \Phi^{xz}\Phi^{yw} + \Phi^{yw}\Phi^{yz}$

End