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Exam on Information Field Theory

17.07.2021

Name: ____________________ Matrikelnummer: ____________________

- The exam consists of **five exercises**. Please do check if you received all of them.
- There are more questions than you will be able to solve within the given time.
- Therefore jump to the next questions if you cannot solve one.
- The working time is **120 minutes**.
- Please provide mathematically conclusive derivations.
- Fill out and hand in the next page together with your solutions
- All solutions need to be handwritten.

Question	**Points**
1	____/11
2	____/15
3	____/17
4	____/18
5	____/13
Bonus	
Total	____/74

Grade	

GOOD LUCK!

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Question 1 ___/11

After the reconstruction of a signal s from data d the posterior signal average is $m = \langle s \rangle_{(s|d)}$ and the uncertainty covariance is $D = \langle (s-m)\,(s-m)^\dagger \rangle_{(s|d)}$. A good quality check of the inferred posterior distribution is to investigate the so called *chi-square statistics* χ^2 and χ^2_m of the *noise weighted data residuals* $\sqrt{N}^{-1}(d - Rs)$:

$$\chi^2 = \mathrm{tr}\langle \sqrt{N}^{-1}(d-Rs)(d-Rs)^\dagger \sqrt{N}^{-1} \rangle_{(s|d)}, \tag{1}$$

$$\chi^2_m = \mathrm{tr}(\sqrt{N}^{-1}(d-Rm)(d-Rm)^\dagger \sqrt{N}^{-1}) \tag{2}$$

Hint: $N = \sqrt{N}\sqrt{N}$

a) Express χ^2 in terms of χ^2_m and remaining terms. ___/6

b) Now – for comparison – calculate the a priori expected χ^2 by averaging over the data realizations for Gaussian prior $\mathcal{G}(s, S)$ and noise distribution $\mathcal{G}(n, N)$ with zero mean, i. e. calculate $\langle \chi^2 \rangle_{(d)}$. Assume the noise to be indepentent of the signal. ___/5

Hint: Start with the definition and use the product rule such that you only need to take one joint expectation value. Your result of part a) is not needed for this part.

Question 2 ___/15

A time-dependent signal $s : \mathbb{R} \to \mathbb{R}$ with stationary Gaussian statistics and a known power spectrum $P_s(\omega) = a\,|\omega|/(\omega^4 + 3\,c^2\,\omega^2)$ is measured via $d_t = s_t + n_t$. The noise also follows stationary Gaussian statistics with a one-over-frequency $(1/f)$ spectrum above some knee-frequency c, i.e., $P_n(\omega) = a\,|\omega|/(4\,c^4 + 2\,c^2\,\omega^2)$. Consider the Wiener filter for this inference problem.

a) Calculate the Fourier-spectra of information propagator and filter function. ___/5

b) Calculate the data convolution kernel $K(x - y)$ of the filter. ___/10

$$m(x) = \int_{-\infty}^{+\infty} \mathrm{d}y K(x-y) d(y), \tag{3}$$

with Wiener filter mean m.

Hint: Use the residue theorem.

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Question 3 ___/17

Let δ_ρ denote the matter overdensity at every position in the local universe, i.e.,

$$\delta_\rho = \rho - \bar{\rho}, \qquad \bar{\rho} = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \rho \, \mathrm{d}^3 x. \tag{4}$$

Assume that the overdensity field is statistically homogeneous and isotropic with a Fourier power spectrum

$$P_{\delta_\rho}(k) \propto k^n. \tag{5}$$

a) Use Laplace's equation for Newtonian gravity, ___/7

$$\Delta\varphi \propto \delta_\rho, \tag{6}$$

to derive the k-dependence of the power spectrum of the gravitational potential, $P_\varphi(k)$

b) Find an expression for the correlation structure of φ in the position basis, ___/10

$$C_\varphi(r) = \langle \varphi(\vec{x})\varphi(\vec{x}+\vec{r}) \rangle_{\mathcal{P}(\varphi)}, \tag{7}$$

and find its scaling with r.

Hint: You might encounter an integral, independent of r, however, to find the scaling you don't necessarily need to solve it.

Question 4 ___/18

A dynamical system $s^t = s(t)$ with $s, t \in \mathbb{R}$ is following the stochastic differential equation

$$\frac{\mathrm{d}s^t}{\mathrm{d}t} + \lambda\, s^t = \xi^t, \tag{8}$$

where $\xi \hookleftarrow \mathcal{P}(\xi) = \mathcal{G}(\xi, \mathbb{1})$ is an unknown excitation field with a Gaussian white-noise prior and $\lambda \in \mathbb{R}$ is a known parameter. The initial system state at $t = 0$ is $s(0) = 0$ and we are concerned with the evolution of the system for $t > 0$ only, which is

$$s^t = \int_0^t dt'\, e^{-\lambda(t-t')} \xi^{t'}. \tag{9}$$

a) For $\lambda = 0$ and $\lambda > 0$ the system has common names. Provide these! ___/1

b) Verify the system evolution provided by Eq. 9. ___/2

c) Calculate the expected first and second moment of the system state, $\overline{s}^t := \langle s^t \rangle_{(\xi)}$ and $S^{tt'} := \langle (s - \overline{s})^t (s - \overline{s})^{t'} \rangle_{(\xi)}$ for the case $0 \le t \le t'$. ___/5

d) Provide the uncertainty dispersion $\sigma^{(\lambda)}(t) = \sqrt{S^{tt}}$ for the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$ and give a short reason why it is decreasing, stagnating, slowly growing, or exponentially exploding for $t \to \infty$ for each of these cases. ___/4

e) Now, a noiseless measurement provides the data $d = s^{t_*}$ for some time $t_* > 0$. Provide a formula for the field posterior mean. ___/3

Hint: The data-space representation of the posterior mean for a Wiener filter measurement situation with $d = Rs + n$ and $\mathcal{P}(s, n) = \mathcal{G}(s, S)\, \mathcal{G}(n, N)$ is $m = S\, R^\dagger \left(R\, S\, R^\dagger + N\right)^{-1} d$.

f) Show explicitely that the data-space representation $D = S - S\, R^\dagger \left(R\, S\, R^\dagger + N\right)^{-1} R\, S$ of the posterior uncertainty covariance and its signal space representation $D = (S^{-1} + R^\dagger N^{-1} R)^{-1}$ are equivalent. ___/3

Question 5 ___/13

The information Hamiltonian of a measurement problem of a real signal field s is

$$\mathcal{H}(d, s) = \frac{1}{2} s^\dagger D^{-1} s - j^\dagger s + \sum_{n=3}^{\infty} \frac{1}{n!} \lambda^\dagger s^n, \tag{10}$$

where D, j, and λ are given.

a) Write down the corresponding log-partition function using Feynman diagrams up to first order in λ and fifth order in D and ignoring any loop diagrams (only tree diagrams). ___/5

b) Give the analytical expression for all the diagrams drawn for the previous step. ___/4

c) Now give a re-summed expression for that part of the full log-partition function that includes all tree diagrams up to first order in λ, but up to <u>any order</u> in D. ___/4