Exam on Information Field Theory

25.07.2017

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- The exam consists of **four exercises**. Please do check if you received all of them.
- There are more questions than you will be able to solve within the given time.
- Therefore jump to the next questions if you cannot solve one.
- The working time is **90 minutes**.
- **No aids** are allowed during the exam.
- In order to gain the full amount of possible points we strictly advise you to use conditional probabilities.

Question	**Points**
1	____/9
2	____/14
3	____/12
4	____/15
Bonus	
Total	____/50

Grade	

GOOD LUCK!

Question 1 ___/9

Assume that a quantity y is linearly dependent on a quantity x, i.e., $y(x) = a + bx$. Assume further that the quantity y has been measured at $m-1$ different positions $(x_i)_i,\ i = 1, \ldots, m-1$, subject to additive uncorrelated Gaussian noise, i.e.,

$$d_i = y(x_i) + n_i, \quad n \hookleftarrow \mathcal{G}(n, N), \quad N_{ij} = \delta_{ij}\sigma_i^2. \tag{1}$$

Assuming a Gaussian prior for the parameters a and b, i.e.,

$$s = \begin{pmatrix} a \\ b \end{pmatrix} \hookleftarrow \mathcal{G}(s, S), \quad S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{2}$$

a linear fit can be performed using Wiener filter theory.

a) Write down the response and posterior covariance for this Wiener filter problem. ___/5

b) You have enough money left to finance one additional measurement with uncertainty σ_m. Let m_a be the Wiener filter estimate after the m measurements. How should you choose the position x_m for that measurement to gain the parameter estimate m_a for a? Use the quadratic loss function $\mathcal{L}(s, x_m) = (a - m_a)^2$ and the formalism of risk minimization. ___/4

Question 2 ___/14

Your prior knowledge about two statistically independent real-valued fields η and ζ is

$$\mathcal{P}(\eta) = \mathcal{G}(\eta, H), \quad \mathcal{P}(\zeta) = \mathcal{G}(\zeta, Z) \tag{3}$$

However, you are only interested in their sum $s = \eta + \zeta$. You have a linear measurement of s with independent Gaussian noise n,

$$d = Rs + n, \quad \mathcal{P}(n) = \mathcal{G}(n, N). \tag{4}$$

a) What is the prior of s? ___/3

<u>Hint</u>: Instead of calculating the prior directly, think which functional form it should have and try to only calculate the quantities needed to specify it.

b) What is the posterior of s? ___/3

c) Calculate the moment generating function $Z_s(j) = \left\langle e^{j^\dagger s} \right\rangle_{\mathcal{P}(s|d)}$. ___/2

<u>Hint</u>: If you could not solve exercise b) you may use $\mathcal{P}(s|d) = \mathcal{G}(s - a - b, C)$.

d) Calculate the one, two and three point correlation functions $\langle s(x) \rangle_{\mathcal{P}(s|d)}$, $\langle s(x)s(y) \rangle_{\mathcal{P}(s|d)}$, and $\langle s(x)s(y)s(z) \rangle_{\mathcal{P}(s|d)}$. ___/6

Question 3 ___/12

Let x be the displacement of a damped harmonic oscillator with randomly fluctuating external force, i.e.,

$$\ddot{x} + \eta\dot{x} + \omega_0^2 x = \alpha\xi, \qquad \xi \hookleftarrow \mathcal{G}(\xi, \mathbb{1}). \tag{5}$$

a) Find the power spectrum $P(\omega)$ of x and sketch it. ___/4

b) Find the singularities of $P(\omega)$. Then set $\eta = 2\omega_0$ and calculate the autocorrelation of x in position space. ___/8
Hint: Use the residue theorem

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi i \sum_k Z_\gamma(a_k)\mathrm{Res}\,(f, a_k) \tag{6}$$

where $Z_\gamma(a_k)$ is the winding number of γ around a_k and $\mathrm{Res}(f, a_k)$ is the residue of f at a_k

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Question 4 ___/15

Consider a real-valued signal field s with a Gaussian prior,

$$\mathcal{P}(s) = \mathcal{G}(s, S), \tag{7}$$

that is observed with an instrument that exhibits an almost linear response,

$$d = R\left(s + rs^2\right) + n. \tag{8}$$

Here, R is a linear operator, $r \in \mathbb{R}$ with $|r| \ll 1$ is a small parameter that determines the strength of the nonlinearity in the instrumental response, s^2 denotes the local squaring of the signal field, i.e., $\left(s^2\right)_x = (s_x)^2$, and n is additive Gaussian noise, i.e.,

$$\mathcal{P}(n) = \mathcal{G}(n, N). \tag{9}$$

a) Consider first the case of an exactly linear response, i.e., $r = 0$. Derive the Hamiltonian ___/2

$$H(d, s) = -\log\left(\mathcal{P}(d, s)\right) \tag{10}$$

for this problem. You may drop all terms that do not depend on s.

b) Show that the posterior probability density in the case with $r = 0$ is of Gaussian form, i.e., $\mathcal{P}(s|d) = \mathcal{G}(s - m_0, D)$, and derive expressions for its mean and covariance, ___/2

$$m_0 = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad D = \left\langle (s - m_0)(s - m_0)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{11}$$

as a function of d, S, N, and R.

c) Now consider the case with small but non-zero r. Calculate the Hamiltonian in this case and write it in the form ___/4

$$H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{\infty} \frac{1}{k!} \Lambda^{(k)}_{x_1 x_2 \cdots x_k} s_{x_1} s_{x_2} \cdots s_{x_k}, \tag{12}$$

where only the coefficients $\Lambda^{(k)}$ depend on r and we use the convention that repeated indices are integrated over. Give expressions for j, D, and all non-zero $\Lambda^{(k)}$. You do not need to calculate H_0.

d) Write down the diagrammatic expansion of the partition function $\log\left(Z(d)\right)$ up to linear order in r. ___/3

e) Find the diagrammatic expressions for the posterior mean and covariance, ___/4

$$m_r = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad \left\langle (s - m_r)(s - m_r)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{13}$$

up to first order in r.