Max Planck Institute
for Astrophysics

# The uncertain uncertainties in the Galactic Faraday sky

**Niels Oppermann**

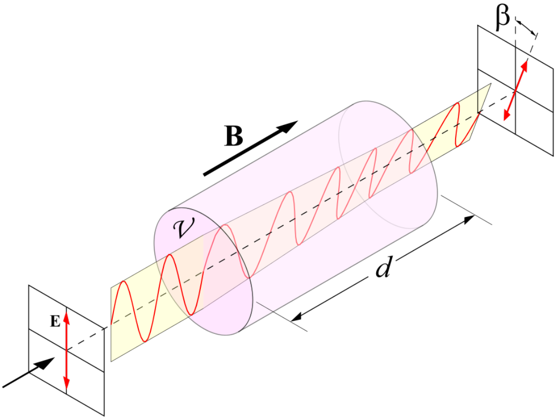
with
G. Robbers, T.A. Enßlin, H. Junklewitz, M.R. Bell, A. Bonafede, R. Braun, J.-A.C. Brown, T.E. Clarke, I.J. Feain, B.M. Gaensler, A. Hammond, L. Harvey-Smith, G. Heald, M. Johnston-Hollitt, U. Klein, P.P. Kronberg, S.A. Mao, N.M. McClure-Griffiths, S.P. O'Sullivan, L. Pratley, T. Robishaw, S. Roy, D.H.F.M. Schnitzeler, C. Sotomayor-Beltran, J. Stevens, J.M. Stil, C. Sunstrum, A. Tanna, A.R. Taylor, and C.L. Van Eck

Bayes Forum, MPE, 2012-05-25

## Outline

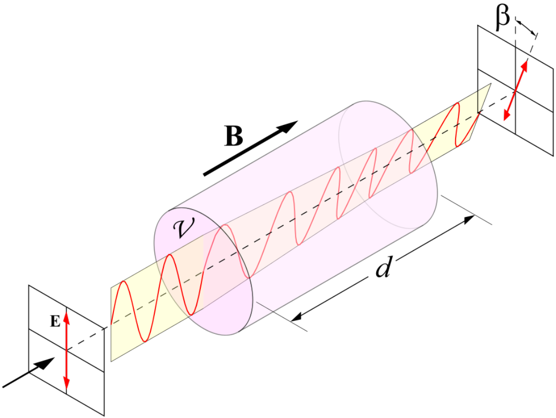
1. **The physics**
   Faraday depth – measuring magnetic fields (sort of)
2. **The statistics**
   priors, marginalizations, and the Gibbs free energy
3. **The application**
   ...of the *extended critical filter* to the Faraday sky
4. **Why we bother**
   if there's time left...

# The Physics



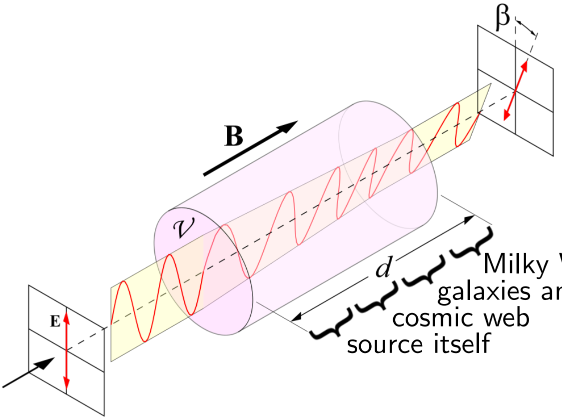

$$\mathrm{d}\beta \propto \lambda^2 n_\mathrm{e}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$

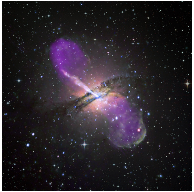
$$\Rightarrow \quad \beta \propto \lambda^2 \int_{r_\mathrm{source}}^{0} n_\mathrm{e}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$




$$\text{Faraday depth: } \phi \propto \int_{r_{\text{source}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$

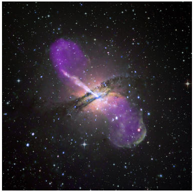
$$\beta = \phi \lambda^2$$

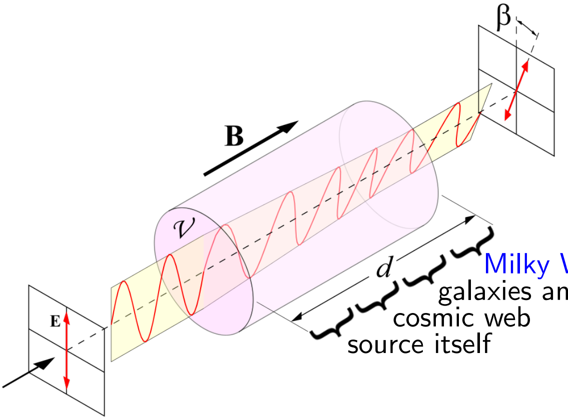


Faraday depth: $\phi \propto \int_{r_{\mathrm{source}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$

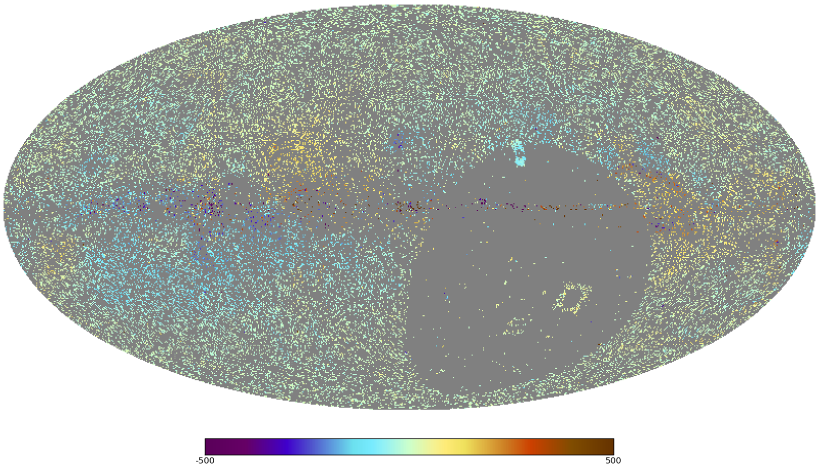
$$\beta = \phi \lambda^2$$



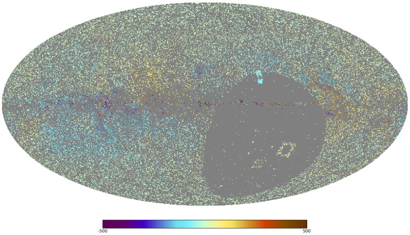



Galactic Faraday depth:

$$\phi \propto \int_{r_{\mathrm{MilkyWay}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$



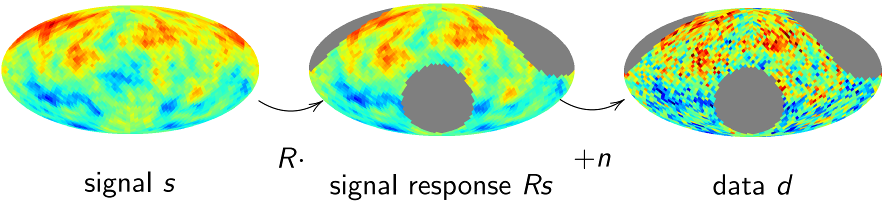

41 330 data points



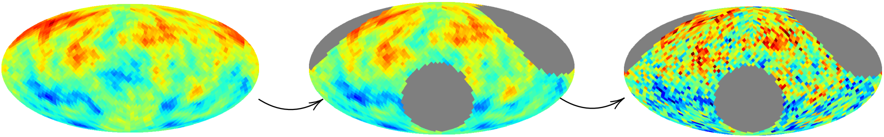

## Challenges

- Regions without data
- Uncertain error bars:
  - complicated observations
  - $n\pi$-ambiguity
  - extragalactic contributions unknown

# The Statistics




$$\boxed{d = Rs + n}$$



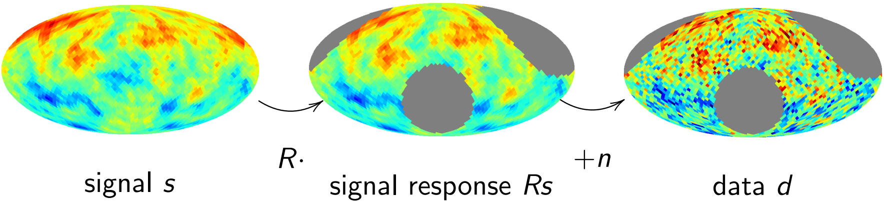
signal $s$ $\quad R\cdot \quad$ signal response $Rs$ $\quad +n \quad$ data $d$

$$\boxed{d = Rs + n}$$

$$\mathcal{P}(s) = \mathcal{G}(s, S)$$

$$\mathcal{P}(n) = \mathcal{G}(n, N)$$

$$\mathcal{G}(s, S) = \frac{1}{|2\pi S|^{1/2}} \exp\left[\frac{1}{2} s^\dagger S^{-1} s\right]$$

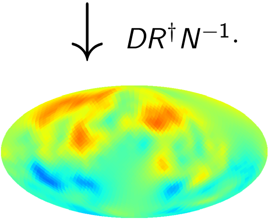



## Wiener Filter

$$m = \int \mathcal{D}s \; s \; \mathcal{P}(s|d)$$

$$d = Rs + n$$

$$m = Dj, \text{ where } \quad \begin{array}{l} j = R^\dagger N^{-1} d \\ D = \left(S^{-1} + R^\dagger N^{-1} R\right)^{-1} \end{array}$$



$$S(\hat{n}, \hat{n}') = \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s)$$

$$\Rightarrow S_{(\ell m),(\ell' m')} = \int \mathcal{D}s \; s_{\ell m} s^*_{\ell' m'} \mathcal{P}(s)$$

$$S(\hat{n}, \hat{n}') = \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s)$$

$$= S(\hat{n} \cdot \hat{n}')$$

$$\Rightarrow S_{(\ell m),(\ell' m')} = \int \mathcal{D}s \; s_{\ell m} s^*_{\ell' m'} \mathcal{P}(s)$$

$$= \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

$\hookrightarrow$ angular power spectrum

$$
\begin{aligned}
S(\hat{n}, \hat{n}') &= \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s) \\
&= S(\hat{n} \cdot \hat{n}') \\
\Rightarrow S_{(\ell m),(\ell' m')} &= \int \mathcal{D}s \; s_{\ell m} s^*_{\ell' m'} \mathcal{P}(s) \\
&= \delta_{\ell \ell'} \delta_{m m'} C_\ell
\end{aligned}
$$

$\hookrightarrow$ angular power spectrum

$$
N_{ij} = \delta_{ij} \sigma_i^2
$$

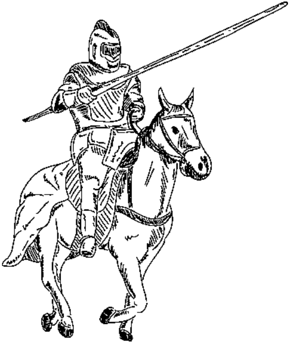
(uncorrelated noise)

$$S_{(\ell m),(\ell' m')} = \delta_{\ell\ell'}\delta_{mm'} C_\ell \qquad N_{ij} = \delta_{ij}\eta_i\sigma_i^2$$

$$S_{(\ell m),(\ell' m')} = \delta_{\ell\ell'}\delta_{mm'} C_\ell \qquad N_{ij} = \delta_{ij}\eta_i\sigma_i^2$$

$$S_{(\ell m),(\ell' m')} = \delta_{\ell\ell'}\delta_{mm'} C_\ell \qquad N_{ij} = \delta_{ij}\eta_i\sigma_i^2$$

Enter the prior

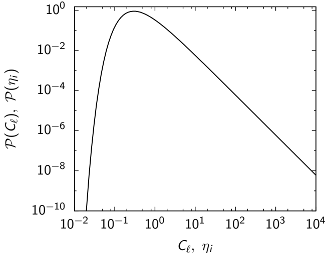


$$S_{(\ell m),(\ell' m')} = \delta_{\ell\ell'}\delta_{mm'} C_\ell \qquad N_{ij} = \delta_{ij}\eta_i\sigma_i^2$$

assume priors for parameters

$$\mathcal{P}\left((C_\ell)_\ell\right) = \prod_\ell \frac{1}{q_\ell \Gamma(\alpha_\ell - 1)} \left(\frac{C_\ell}{q_\ell}\right)^{-\alpha_\ell} \exp\left(-\frac{q_\ell}{C_\ell}\right)$$

$$\mathcal{P}\left((\eta_i)_i\right) = \prod_i \frac{1}{q_i \Gamma(\alpha_i - 1)} \left(\frac{\eta_i}{q_i}\right)^{-\alpha_i} \exp\left(-\frac{q_i}{\eta_i}\right)$$




$\Rightarrow$ marginalize over all possible parameters

$$S_{(\ell m),(\ell' m')} = \delta_{\ell\ell'}\delta_{mm'} C_\ell \qquad N_{ij} = \delta_{ij}\eta_i\sigma_i^2$$

assume priors for parameters

$$\mathcal{P}\left((C_\ell)_\ell\right) = \prod_\ell \frac{1}{q_\ell \Gamma(\alpha_\ell - 1)} \left(\frac{C_\ell}{q_\ell}\right)^{-\alpha_\ell} \exp\left(-\frac{q_\ell}{C_\ell}\right)$$

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$\Rightarrow$ marginalize over all possible parameters

Problem: $\mathcal{P}(s|d)$ is non-Gaussian.
Solution: Find Gaussian $\mathcal{G}(s-m, D)$, that best approximates $\mathcal{P}(s|d)$.

- Minimize Kullback-Leibler divergence

$$d_{\mathrm{KL}} = \int \mathcal{D}s\, \mathcal{G}(s-m, D)\, \log\left(\frac{\mathcal{G}(s-m, D)}{\mathcal{P}(s|d)}\right)$$

or

- Minimize approximate Gibbs free energy

$$G = \left\langle H_{\mathcal{P}(s|d)} + \log\left(\mathcal{G}(s-m, D)\right)\right\rangle_{\mathcal{G}(s-m,D)}$$

Enßlin & Weig (2010)

Problem: $\mathcal{P}(s|d)$ is non-Gaussian.
Solution: Find Gaussian $\mathcal{G}(s-m, D)$, that best approximates $\mathcal{P}(s|d)$.

## Extended Critical Filter

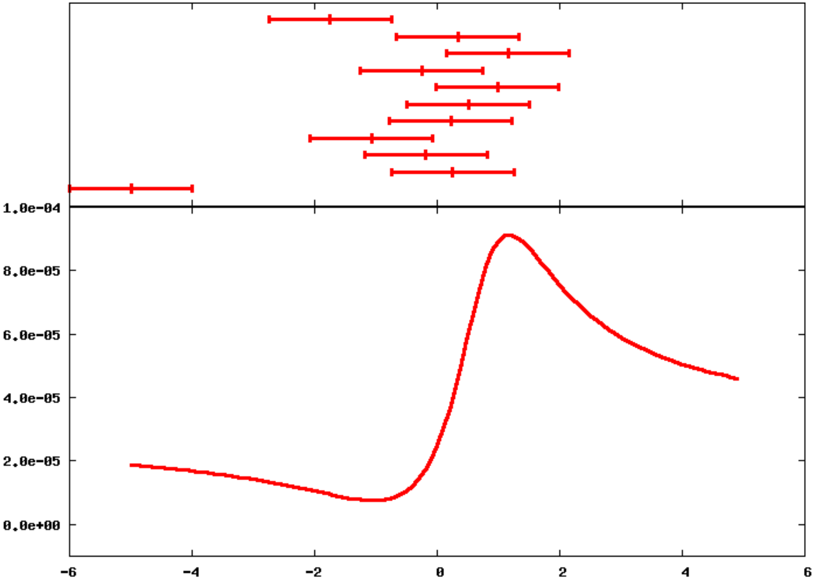
$$m = Dj, \quad D = \left[\sum_\ell C_\ell^{-1} S_\ell^{-1} + \sum_i \eta_i^{-1} R^\dagger N_i^{-1} R\right]^{-1},$$

$$j = \sum_i \eta_i^{-1} R^\dagger N_i^{-1} d$$

$$C_\ell = \frac{1}{\alpha_\ell + \ell - 1/2}\left[q_\ell + \frac{1}{2}\mathrm{tr}\left(\left(mm^\dagger + D\right) S_\ell^{-1}\right)\right]$$

$$\eta_i = \frac{1}{\alpha_i}\left[q_i + \frac{1}{2}\mathrm{tr}\left(\left((d - Rm)(d - Rm)^\dagger + D\right) N_i^{-1}\right)\right]$$

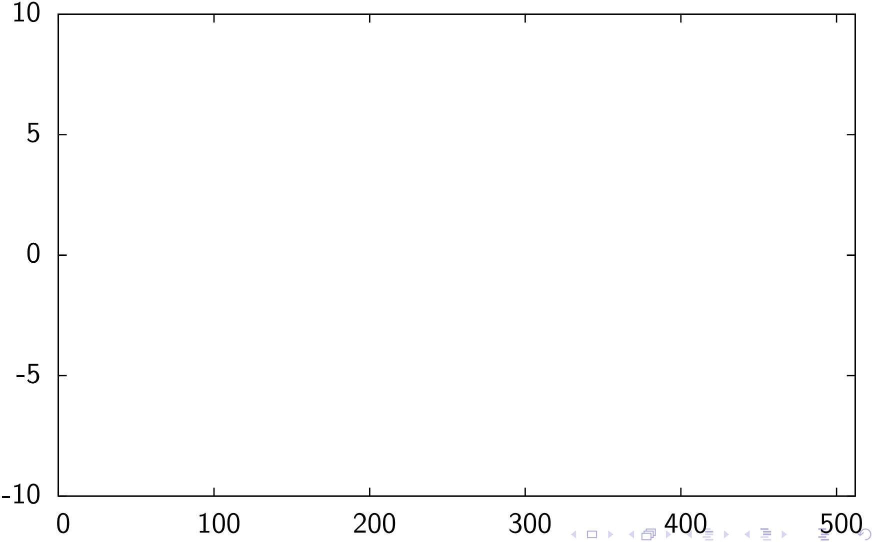
# 0D test case



# 1D test case

**Assumptions:**

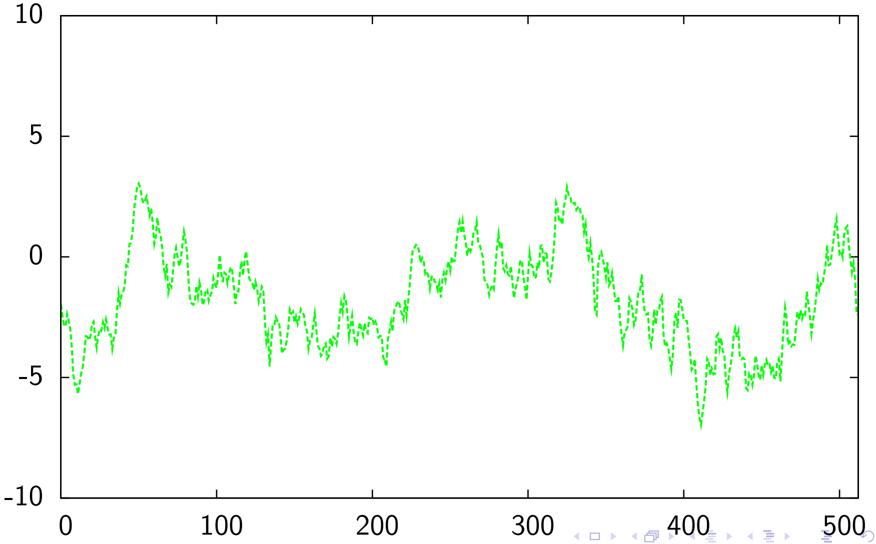
- 
- 



# 1D test case

**Assumptions:**

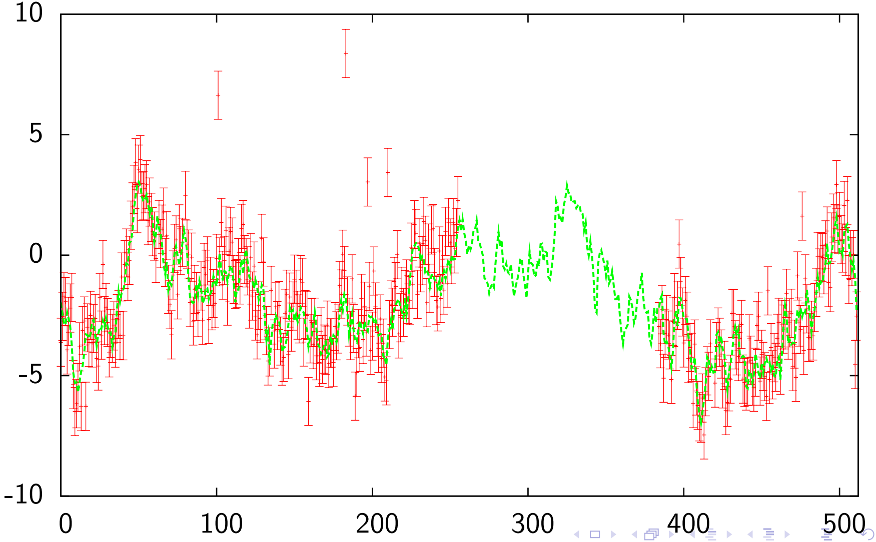
- signal field statistically homogeneous Gaussian random field
- 



# 1D test case

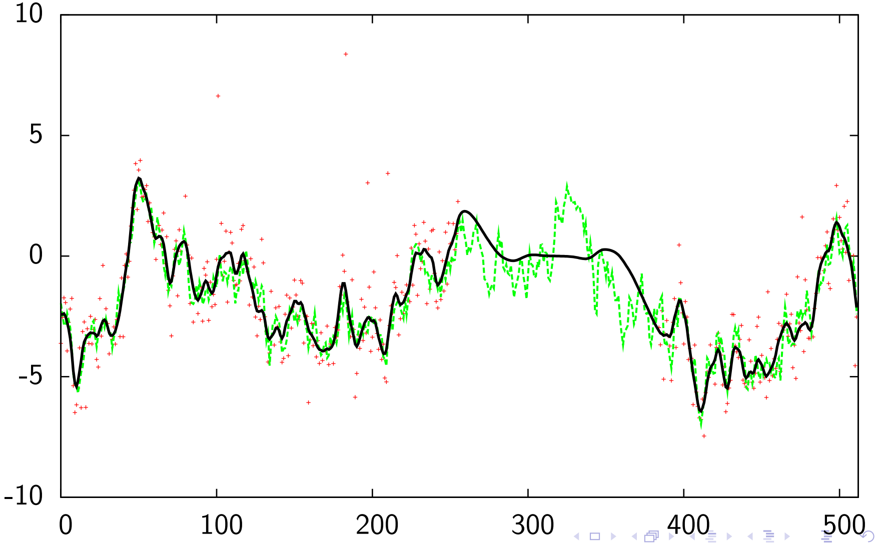
**Assumptions:**

- signal field statistically homogeneous Gaussian random field
- noise uncorrelated, Gaussian



# 1D test case

- Reconstruct (iteratively):
  signal, power spectrum, noise variance



# 1D test case

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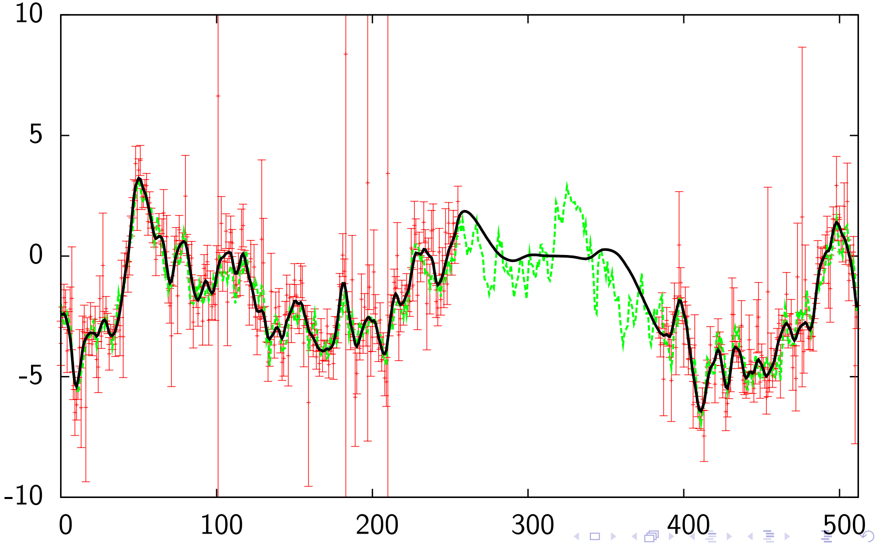
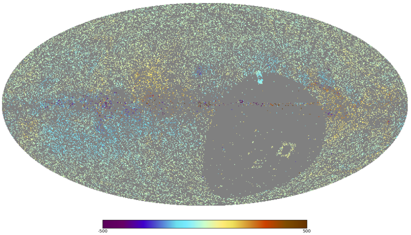
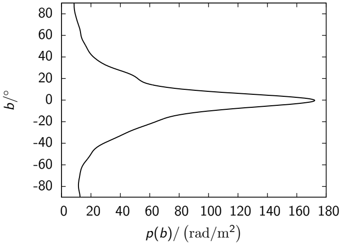


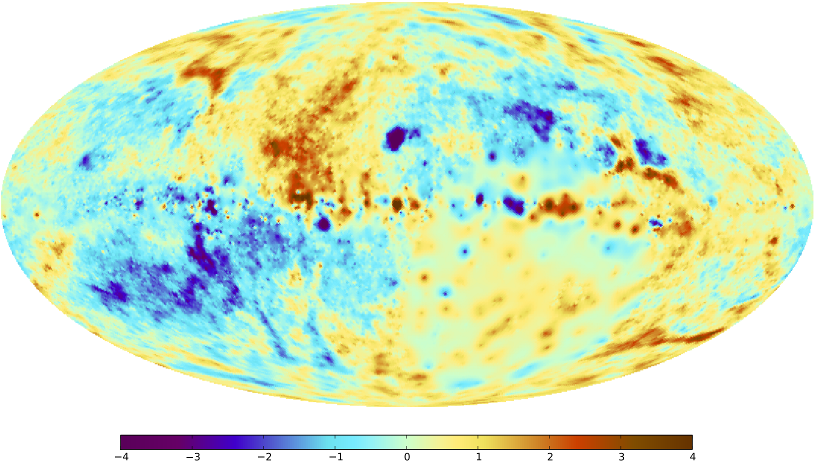
# The Application



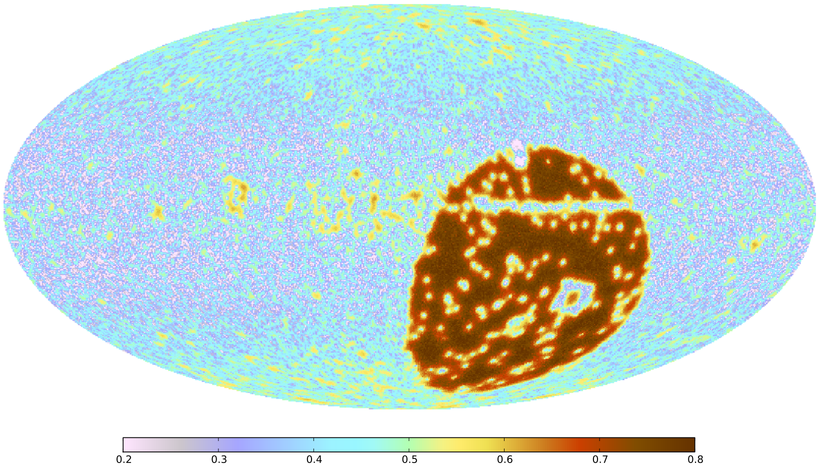

- Approximate $s(b,l) := \frac{\phi(b,l)}{p(b)}$ as a statistically isotropic Gaussian field
- $R$: multiplication with $p(b)$ and projection on directions of sources
- $N_{ij} = \delta_{ij}\eta_i\sigma_i^2$

# posterior mean of the signal



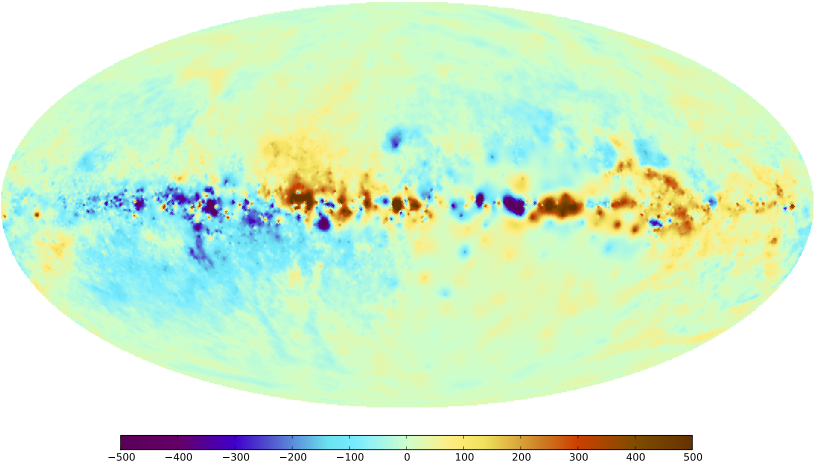

$m$

# uncertainty of the signal map



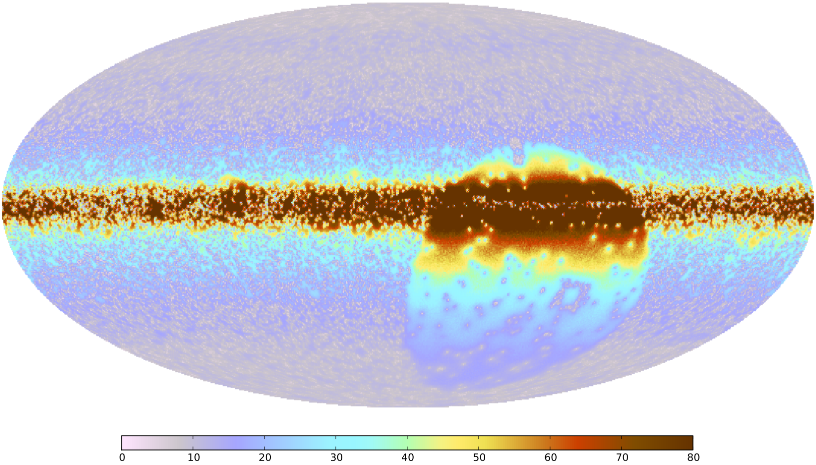

$$\sqrt{\mathrm{diag}\left(D\right)}$$

## posterior mean of the Faraday depth



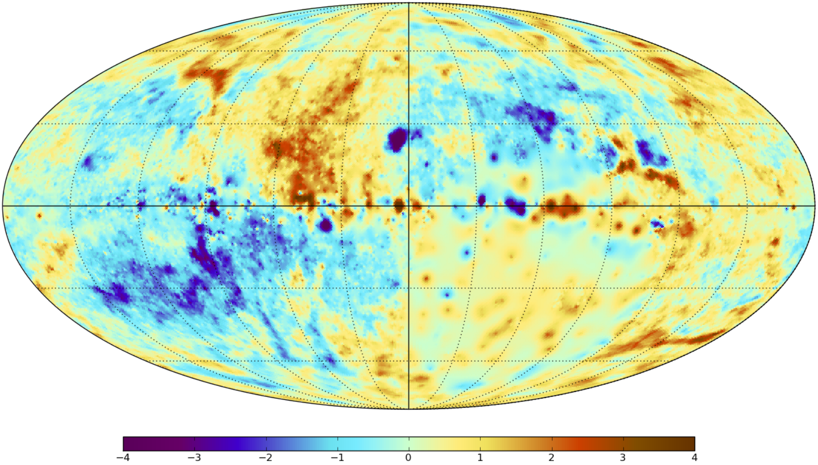

*pm*

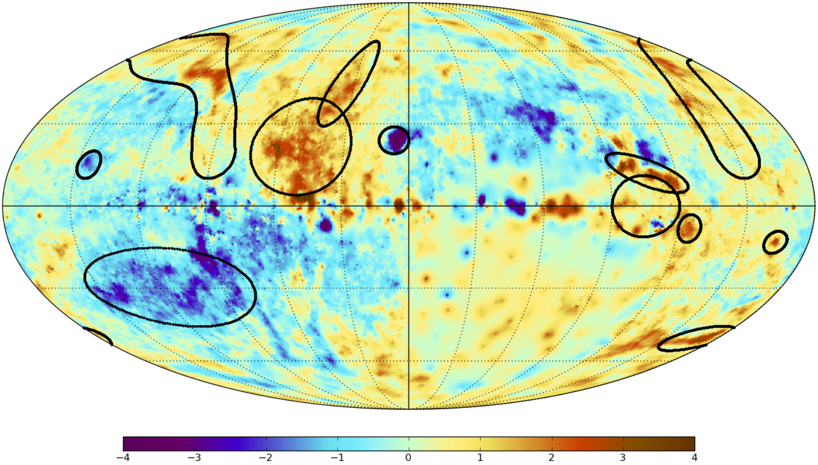
## uncertainty of the Faraday depth

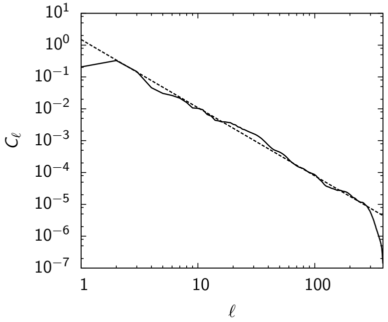


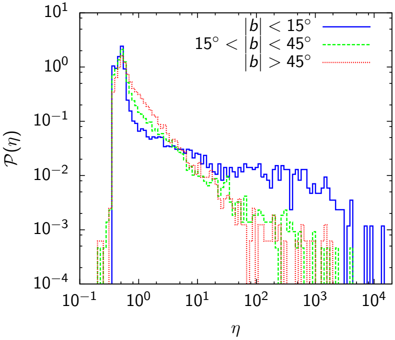

$$p\sqrt{\mathrm{diag}\,(D)}$$

# Why we bother


−4
−3
−2
−1
0
1
2
3
4

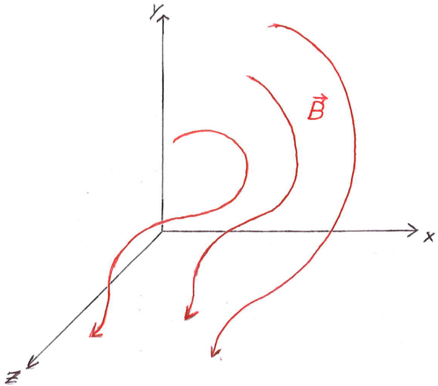

−4
−3
−2
−1
0
1
2
3
4



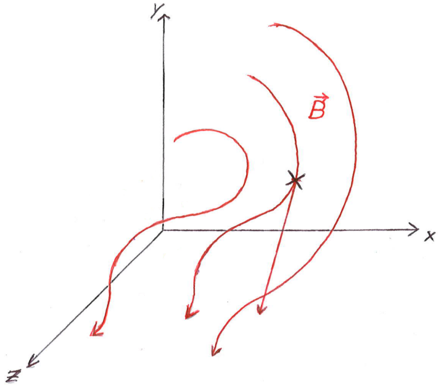




$$C_\ell \propto \ell^{-2.14}$$

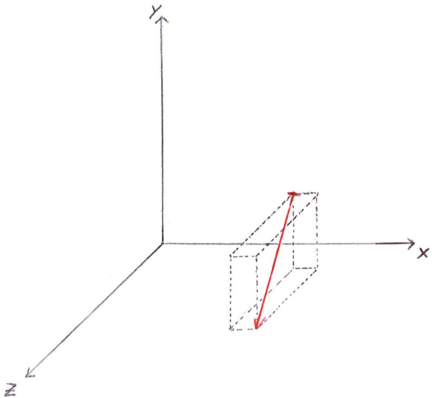
$$N_{ij} = \langle n_i n_j \rangle = \delta_{ij} \eta_i \sigma_i^2$$

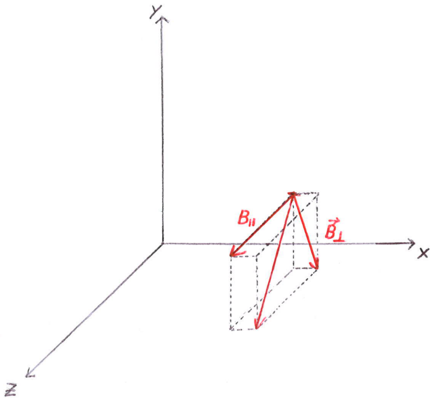


y
x
z
$\vec{B}$

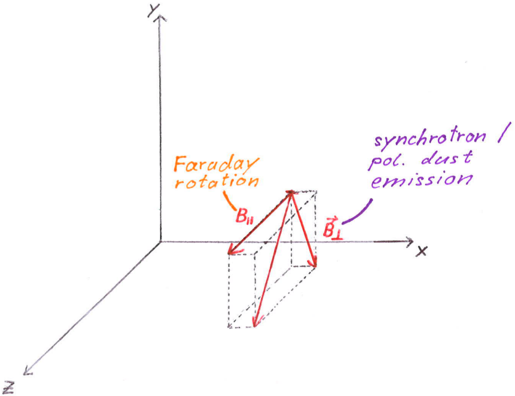


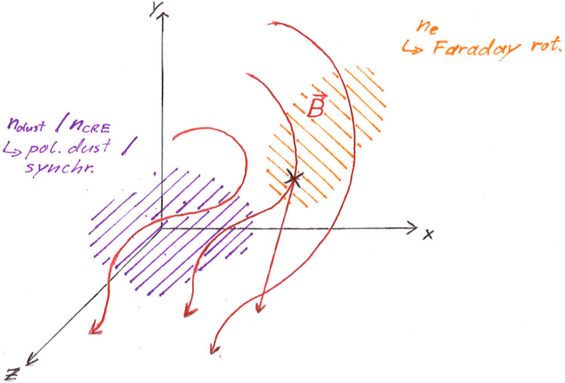
y
x
z
$\vec{B}$


Y
X
Z



Y
X
Z
$B_{\parallel}$
$\vec{B}_{\perp}$


Y
X
Z
Faraday rotation
B∥
B⊥
synchrotron / pol. dust emission



y
x
z
$n_e$
↳ Faraday rot.
$\vec{B}$
$n_{dust}$ / $n_{CRE}$
↳ pol. dust /
synchr.

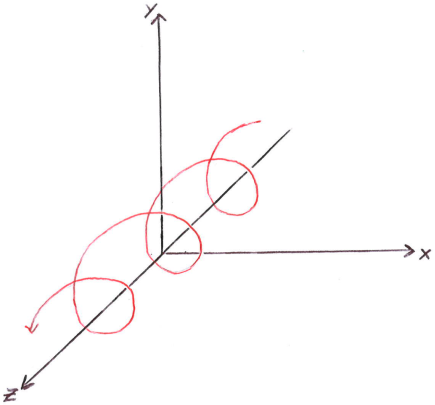
## Summary

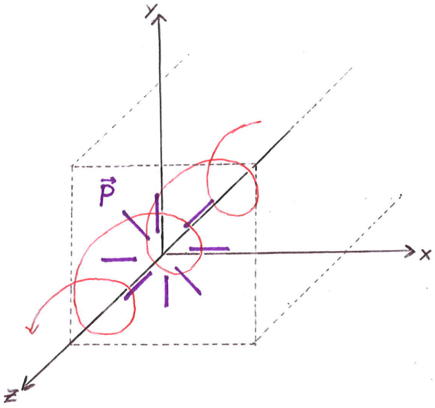
1. The *extended critical filter* reconstructs
   - "smooth" signals
   - from data that are
     - noisy
     - and incomplete.
2. It makes use of the
   - signal covariance
   - and noise covariance

   even though they are unknown.
3. Only the eigenvectors of these matrices have to be known.

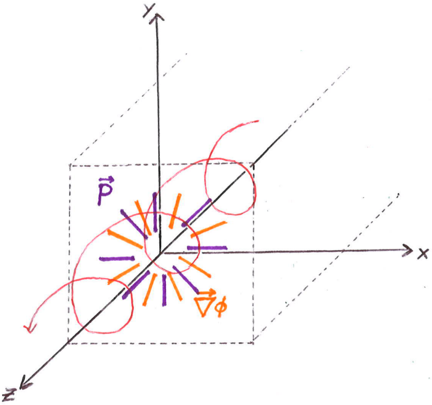
`http://www.mpa-garching.mpg.de/ift/faraday/`

# Backup


y
x
z



y
x
z
$\vec{P}$




known as **LITMUS** procedure

Junklewitz et al. 2011A&A...530A..88J
Oppermann et al. 2011A&A...530A..89O